THE THREE GREAT PROBLEMS OF ANTIQUITY, CONSIDERED IN THE LIGHT OF MODERN MATHEMATICAL RESEARCH.

Vorträge über ausgewählte Fragen der Elementargeometrie. F. KLEIN. Ausgearbeitet von F. TÄGERT. Leipzig, Teubner, 1895. pp. 66.

AMONG the minor mathematical works published during the past year, one of the most interesting is Klein's Festschrift for the third meeting of the association for the advancement of mathematical and scientific teaching in the Gymnasia. The author has himself explained, in his paper *Ueber Arithmetisirung der Mathematik*, that his main object in writing this pamphlet was to emphasize the necessity for strict logical developments as a corrective to the tendency to rely too exclusively on intuitive proofs. Intended primarily for teachers of the more elementary mathematics, this pamphlet is confined to the three great problems of antiquity, as they appear in the light of modern research—or rather to the mathematical investigations for which these problems have furnished the text. These problems, (1) the duplication of the cube, (2) the trisection of an angle, (3) the quadrature of the circle, presented themselves at a very early stage in the development of mathematics, and they naturally present themselves correspondingly early in the mathematical development of the individual.

The Greek mathematicians, striving to solve these problems—problems for each of which the solution was impossible, within the domain of the geometry of the straight line and circle—were led to investigate and discover nearly all that lay within their originally unconsciously imposed boundaries. "Let it be granted, that a straight line may be drawn from any one point to any other point," "let it be granted, that a circle may be described from any centre, at any distance from that centre," these indicate the limitations of elementary geometry as understood by the Greeks, limitations recognized and formulated in the brightest period of Greek geometry by Euclid. It has been left to later times to recognize that other not less important limitations are implied in the definitions and axioms.

Still striving after the solution of these problems, Archytas, Eudoxus and their successors were led to enlarge their

boundaries, admitting other methods, in the first instance the construction of conics, by which they solved the problems that they classed as "solid," and then the construction of higher curves, "linear loci," for use in treating those questions that were neither plane nor solid.

In this pamphlet the essential nature of the problems is sharply defined; the domain of Greek geometry is investigated, as to its contents and its boundaries; the correspondence with different parts of algebra is traced, and it is shown, in part all too briefly, how the successively enlarged domain of algebra corresponds with the different parts of geometry, the admissible algebraic operations being represented by the postulated geometrical constructions. Rational algebraic operations are represented by linear constructions; operations depending on the extraction of the square root can all be performed by means of straight lines and circles, in fact, by means of straight lines and a single circle, a possibility shown by Poncelet. These operations can also be performed by means of circles only, for Mascheroni wrote a book on geometry in which the only admissible construction is that of a circle, with any centre, passing through any assigned point. Klein refers to Hutt for an account of this book; a paper by Cayley in the *Messenger of Mathematics*, vol. 14, p. 179, may be more available for some readers. This paper gives sufficient account of the work to enable one to see the course of the proofs; the constructions depend chiefly on the conception of symmetry, points on a line being determined by the intersections of equal circles whose centres are symmetrically placed with respect to the line.

The pamphlet is divided into two parts, the first of which deals with algebraic numbers, the second with transcendental numbers. In the first part, Ch. I. explains the nature of a system of conjugate irrational quantities, and proves that there exists only one irreducible equation satisfied by one, and consequently by all, of the quantities of any such system. The chapter contains also a discussion of the nature of the algebraic equations that can be solved by the extraction of square roots; this discussion may be described as elementary, inasmuch as all the steps are minutely explained, but it gives the rigorous proof of the important theorem that an irreducible equation cannot be solved by the extraction of square roots, and hence cannot be constructed by means of straight lines and circles, unless its degree be a power of 2. In order to solve a cubic equation the domain of algebra must be enlarged by adjoining the operation of taking the cube root; hence problems whose

algebraic expression leads to an equation of the third degree cannot be solved by elementary geometry. In Ch. II. it is shown that the first two of the three problems under discussion are of this type; they are therefore outside the domain of elementary geometry.

Since in Ch. I. the possibility of solving an apparently insoluble equation, if the degree be a power of 2, has presented itself, in Ch. III. this question is considered more carefully. It is not possible (*i. e.*, within the assigned limits of construction) to trisect *any* angle, but we can trisect a complete angle, we can divide the circumference of a circle into three equal parts. Can we divide the circumference into *any* number of equal parts? This inquiry opens out an extensive field of modern research, and may well serve to introduce the would-be mathematician to some comprehension of what modern mathematics means. The division into 3, 4 or 5 equal parts, with the bisection of each of these parts any number of times, gives the extent of the Greek achievement in this line, and was supposed to give the limit of all possible achievement. The list was however extended by Gauss, who showed that the division can be performed for any prime number of the form $2^{2^\mu}+1$, and for any number whose prime factors other than 2 are of this form, and non-repeated. Moreover, these numbers give all for which the division is possible.

In the first place, assuming for a moment that the prime must be of the form $2^\lambda+1$, it is shown that a necessary but not sufficient condition that this represent a prime is $\lambda =$ a power of 2, 2^μ. It is remarked that the values 0, 1, 2, 3, 4, for μ give prime numbers 3, 5, 17, 257, 65537; that the values 5, 6, 7 for μ do not give prime numbers; and that the investigation has not been carried any further. At this point Klein suggests that possibly 4 is the highest value for μ that gives a prime. In the *Récréations mathématiques* of E. Lucas, however, the remark is made (vol. II., p. 235) that Eisenstein enunciated the theorem, "The number of primes of the form $2^{2^\mu}+1$ is infinite," giving no proof, though Lucas suggests that possibly he had one. The theorem may of course be purely conjectural.

It is now to be shown that the equation $\dfrac{z^p-1}{z-1}=0$,

i. e. $$z^{p-1}+z^{p-2}+\cdots+z+1=0,$$

which with the ordinary use of the complex variable is the equation for p-section of a circle of unit radius, is irredu-

cible. This, by the Gauss lemma, is made to depend on the proof that the expression cannot be resolved into factors with integral coefficients. Here Klein follows Eisenstein's method, which, while perfectly simple, is not satisfactory in point of form, for the proof is made to depend on the somewhat irrelevant transformation $z=x+1$, by which it is shown that the contrary supposition is untenable. When it has been shown that the equation is irreducible the rather lengthy argument of Ch. I. shows that the degree of the equation, *i. e.*, $p-1$, must be a power of 2.

As regards the actual theorem, a different proof is to be found in the December number of the BULLETIN, in the paper by Dr. James Pierpont. This paper deals with the complete theorem under consideration, giving an admirably simple proof, which sets the essential part of the theorem in evidence. It is to be shown that the equation

$$F(x)=x^{(p-1)p^{\alpha-1}}+\cdots+1=0,$$

where m, the number of parts into which the circle is to be divided, $=p^{\alpha}$, cannot be rationally resolved; this is accomplished by following a method of Kronecker, preferable to that of Eisenstein, and presenting no greater difficulty; it is then shown that considering irrational factors of $F(x)$, the irrationalities depending only on the extraction of a square root, the only possible way in which $F(x)$ can split up the first time is into two conjugate factors of equal degree. Applying the theorem to each of these, and then to their factors, and so on, it is seen that the degree of F must be a power of 2.

The theorem having been established, the solution of the cyclotomic equation follows. It is possible to divide the roots into two periods whose sum and product are integral numbers; hence the two periods are the roots of a quadratic equation with integral coefficients. Taking one of the roots of this quadratic (that is, one of the two periods first formed), this can again be divided into two periods, given as the roots of a quadratic whose coefficients are rational in the roots of the first quadratic. In this way we obtain the solution of the cyclotomic equation by the solution of a chain of quadratics, the coefficients in any one of which depend on the roots of the preceding one. The process is fully explained in Ch. IV., being applied to the equation

$$z^{16}+z^{15}+\cdots+z+1=0,$$

from the solution of which there is obtained a linear con-

struction for a regular polygon of 17 sides in a given circle.* The equations are so carefully worked out and so minutely explained that the chapter will throw much light on the difficulties that some readers will find in the first few pages of the paper. But seeing how much importance attaches to the fact that the equation is irreducible until the domain of rationality is extended, we could wish that Klein had thought it worth while to give the successive factors;

$$z^{16}+z^{15}+\cdots+1=\Big(z^8-\eta_0 z^7+(2-\eta_0)z^6+(3-\eta_0)z^5+(1-2\eta_0)z^4+$$
$$(3-\eta_0)z^3+(2-\eta_0)z^2-\eta_0 z+1\Big)\Big(z^8-\eta_1 z^7 \text{ etc.},\cdots\Big),$$

$$z^8-\eta_0 z^7+\cdots+1=\Big(z^4-\eta_0' z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0' z+1\Big)\Big(z^4-\eta_1' z^3$$
$$+\frac{\eta_1'\eta_0-\eta_0'+1}{2}z^2-\eta_1' z+1\Big),$$

$$z^4-\eta_0' z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0' z+1=\Big(z^2-\eta_0'' z+1\Big)\Big(z^2-\eta_1'' z+1\Big);$$

this would have thrown some additional light on the significance of the early part of the paper.

Chapter V. indicates briefly the points of interest in connection with the construction of higher equations. The cubic and biquadratic are solved by means of conics, and thus conics are needed for the solution of problems whose algebraic expression leads to such equations. An example that naturally presents itself is the construction of the regular heptagon by means of a parabola and a rectangular hyperbola; and it is at once perceived that if conics are admitted, many more regular polygons can be described. Apparently Descartes knew that the regular heptagon and nonagon can be described by conics, for when discussing constructions by curves of higher order, he says that these can be applied to construct regular polygons of 11 or 13 sides. Now the regular polygons with 3, 4, 5, 6, 8, 10, 12

* In this connection it is perhaps worth while drawing attention to the very simple construction for the regular 17-ic given by Mr. H. W. Richmond in the *Quarterly Journal*, vol. 26, p. 206, 1893. The construction is simple in practice; theoretically it is not as simple as the one here given, for the equations are combined in such a way as to require circles in their construction. It might probably with slight modifications give a Mascheroni construction, which Klein remarks (p. 27) has not yet been considered for this case.

sides were described by the Greeks, and supposing Descartes to be aware that for 7 and 9 sides only conics are needed, his idea was apparently that the first regular polygons for whose construction higher curves are required are those of 11 and 13 sides. (The 13-ic can however be described by conics.) The omission of any explicit mention of the 7-ic and 9-ic proves nothing in the case of Descartes, for at the point where he might naturally have given it he had already devoted several pages to the construction of cubic equations, and was probably, as he himself says in another place with less reason, "bored already with writing so much about it." Such omissions may surely be forgiven for the sake of his concluding remark: "Et j'espère que nos neveux me sauront gré, non seulement des choses que j'ai ici expliquées, mais aussi de celles que j'ai omises volontairement, afin de leur laisser le plaisir de les inventer."

The first part of the paper has made clear the meaning of algebraic numbers, irrationalities successively introduced in the solution of algebraic equations. For the discussion of the third great problem, the quadrature of the circle, it is necessary to show that algebraic numbers form only a part, in point of fact an insignificant part, of the numbers that claim consideration. The chapter devoted to this cannot fail to interest many hitherto unfamiliar with Cantor's work. It is first to be shown that there is a (1, 1) correspondence between positive integers and all real algebraic numbers, that is to say, that the real algebraic numbers can be counted. To prove this it is shown, by a particular grouping of the irreducible equations whose roots give the algebraic numbers, that the numbers can be arranged in a singly infinite series of groups, those contained in any one group being finite in number; and that thus all the real algebraic numbers, positive and negative, can be arranged in an absolutely determinate order, which is not the order of magnitude; and so can be spoken of as 1st, 2nd, 3rd, etc., that is, put in (1, 1) correspondence with the positive integers.

The next step is to show how numbers can be constructed that shall not be contained in this orderly series. The number being required to lie within certain limits, so that there are given a certain number of decimal places, *e. g.*, 5, the digits in the following places have to be selected so that the number differs from all of the series. For a reason explained, the digit 9 is avoided. The 6th digit is chosen to be different from the 6th of the first algebraic number, and thus the number constructed will certainly be different from this;

the 7th digit is chosen to be different from the 7th of the second algebraic number, by which we ensure that the number written down is not the second, and so on. Hence we are assured of the existence of numbers that are not the roots of any algebraic equation; that is, the existence of transcendental numbers is proved, and it is shown how they can be written down. Moreover, since the choice of the digit to be written in any assigned place is restricted only by the exclusion of two digits, 9 and one other, we may choose any one of the 8 that are left, zero being admissible in the same way as any other. Hence between any two algebraic numbers there are 8^{∞} transcendental numbers, (not ∞^{8} as stated in the pamphlet,) and real algebraic numbers form only a small part of all numbers.

The possibility of determining whether a given number is transcendental or not cannot be considered generally. A terminating or circulating decimal is obviously algebraic; a non-terminating and non-circulating decimal obviously cannot be given in full. Transcendental numbers can therefore only be given by some law of formation, as is seen in the case of e and π; but since algebraic numbers can also be given in this way, the fact that a number is so given proves nothing. The impossibility of giving any general criterion is brought home by the nature of the proofs for e and π. The connection of the two numbers, exhibited in the fundamental equation $e^{i\pi}+1=0$, gives the transcendency of π as a corollary from the theorem:—The quantity e is not a root of any equation in which both coefficients and exponents are algebraic numbers. To the consideration of e and π chapters III. and IV. are devoted; these give the most recent proofs, published as late as 1893.

Since the number π is transcendental it follows that no algebraic curve can be found in which the relation of derived straight lines is that of the circumference of a circle to the diameter. Hence the quadrature of the circle, the third great problem of antiquity, cannot be solved by means of algebraic curves.

Thus in these few pages some of the most striking results of modern mathematics are made accessible to many who would otherwise hardly have heard of them. But while reading this brilliant exposition it is difficult to avoid cherishing a lurking regret, which is possibly very ungracious, that Klein could not himself spare time to arrange his work for publication; for though we have here in full measure the incisive thought and cultured presentation which together make even strict logic seem intuitive, yet at times we miss

the minute finish and careful proportion of parts that we feel justified in expecting from him. And yet revision and consolidation might have seriously interfered with the graphic simplicity of these chapters, and left them less adapted to their special purpose. Any English-speaking association with aims similar to those of the German association for which the pamphlet was prepared would do a service in publishing a thoroughly good translation of this inspiring work and circulating it as widely as possible.

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January 15, 1896.

PAINLEVÉ'S LECTURES ON DYNAMICS.

I. *Leçons sur l'intégration des equations différentielles de la Mécanique et Applications.* Par P. PAINLEVÉ. pp. 291; 4to (lithographed).

II. *Leçons sur le Frottement.* Par P. PAINLEVÉ. pp. VIII.+ 111; 4to (lithographed). Paris, A. Hermann, 1895.

The publication of Mr. Painlevé's lectures on the integration of the differential equations of dynamics will be welcomed by everyone interested in the progress of theoretical mechanics. It is a long time since Jacobi's *Vorlesungen über Dynamik* appeared, and a strong need was felt of a systematic work which would contain the more recent researches in this important branch of mathematical science. Mr. Painlevé has admirably supplied this need. The lectures are not intended for beginners in theoretical mechanics, but rather as a supplementary course for those already familiar with its elements.

The first three lectures contain the fundamental definitions and principles of dynamics, the propositions relating to the first integrals of the motion of rigid systems and the theory of the motion of a solid body, the whole followed by a number of examples. Despite the brevity of this exposition it is clear and instructive, owing to several interesting remarks.

The fourth lecture deals with the general equations of the motion of systems. Lagrange's equations with the multipliers λ (also called Lagrange's equations of the first form) are derived from the consideration of the virtual work, which leads the author to a classification of sys-

tems into those with friction and without. In the next lecture this distinction is further discussed and a proof is given of Gauss's theorem that the constraint of a system is minimum when it is one without friction. Then the motion of systems with friction is studied and applied to the motion of a point on a surface. However the general theory of such systems constitutes the subject of a more detailed study in another of Mr. Painlevé's courses to which we will return later.

In the next two lectures, after establishing d'Alembert's equation and the principle of virtual velocities, the author passes to Lagrange's equations (of the second form), concluding with an enumeration of the principal systems without friction. All following lectures are restricted to such systems only.

Lectures VII.–IX. deal with the applications of Lagrange's equations, the cases being discussed separately when the equations of constraint do or do not contain the time explicitly. The author further proves Liouville's theorem relating to a large class of systems the motion of which can be found by quadratures. Several applications of this theorem are given. The classic problem of the motion of a point attracted by two fixed points and a number of other well chosen examples are treated in the course of these three lectures, the solutions being given with great simplicity and elegance.

Bour's differential equations of relative motion in the second Lagrangian form with examples taken from Gilbert's interesting memoir, on the applications of Lagrange's equations to the problem of relative motion, are the subject of the tenth lecture.

The next lecture begins with Lejeune-Dirichlet's theorem about the stability of the equilibrium of a system after which the author applies Lagrange's equations to the study of small oscillations of systems.

The remaining six lectures (XII.–XVII.) deal with the integration of the differential equations of dynamics principally after the works of Hamilton and Jacobi. The subjects of lectures XII. and XIII. are the canonic equations and Jacobi's theory of the last multiplier. Bour's canonic equations for relative motion are indicated but not discussed. The theory of the last multiplier is first applied to the motion of rigid systems in general. The application of the same theory to the canonic equations constitutes the subject of the fourteenth lecture, which is illustrated by a number of examples.

The next lecture contains Hamilton-Jacobi's method of integrating the differential equations of dynamics and the several simplifications which may occur in practice are discussed. A new demonstration is given of Liouville's theorem together with a generalization of the same due to P. Stäckel.

The sixteenth lecture presents the greatest novelty and interest. It deals with the more modern researches, the impulse to which was given first by Bertrand's principle of similitude in mechanics and quite recently by the remarkable paper of Stäckel (Crelle 1891) on the analytic equivalence of dynamic problems. The reader will find in this lecture a good deal of Mr. Painlevé's own researches chiefly contained in his important memoir on the transformation of the equations of Dynamics (J. de Liouville 1894). In the beginning of this lecture the author proves that the paths of a system depend in general on $2k-1$ arbitrary constants, k being the degree of freedom of the system, and he gives the condition necessary and sufficient in order that this number reduce to $2k-2$. The case is considered next, when these paths coincide with the geodesics. Then, after a discussion of the general case when the acting forces are independent of the velocities, the author passes to a detailed study of the real paths of a system; the principle of least action and Hamilton's principle are touched upon in the course of this lecture.

The last lecture relates to conservative systems. After establishing Poisson's theorem, the author returns once more to Hamilton-Jacobi's canonic equations and gives a concise but clear exposition of Jacobi and Mayer's methods for the integration of partial differential equations of the first order, concluding with the application to Jacobi's equation of Legendre's transformation, which is a particular case of Jacobi and Lie's transformations of contact.

It is to be regretted that the author did not add a chapter on the theory of perturbations which is so closely connected with the subject of his lectures. However, if these lithographed pages in which the exposition was necessarily limited are only the precursor of a more complete printed course, we may hope that the theory of perturbations will still find its place among Mr. Painleve's interesting lectures.

We now pass to the second of Mr. Painlevé's courses. The *lectures on friction* are a valuable contribution to rational mechanics. The author proposes to solve the following problem: *to develop for systems with friction a general theory similar to the one existing for systems without friction*, as far as

the empiric laws of friction allow such a general treatment. The lectures before us deal first with discontinuous systems, and then with such continuous systems which can be considered as formed of infinitesimal elements that remain unchanged throughout the motion,* the other systems (to which belong extensible strings and membranes, compressible fluids and so on) being reserved by the author for another course to appear later. It must be remarked moreover that all cases where the conditions of constraint to which the systems are subject are over-abundant were systematically avoided by the author, since the motion of such systems in general depends on their interior constitution and belongs therefore to the theory of elasticity.

After recalling his general definition of systems with friction given in the fourth lecture on the integration of the differential equation of dynamics, derived from the consideration of the virtual work, the author examines first two special cases, namely, the motion of a point on a curve and on a surface, then passing to the general form of the laws of friction. We arrive here at a most interesting part of this course, both for its novelty and importance. The notion of partial friction is introduced and the author gives a detailed study of the laws of friction which result from the combination of the laws of partial friction (*combinaison des liaisons*). An important result is obtained when the conditions of constraint can be grouped into two parts: one, G_1, with friction and the other G_2, without. For, in such cases it is advantageous not to introduce the reactions due to the group G_1; to this purpose it suffices to form Lagrange's differential equations with regard to the parameters which would define the position of the system if only the conditions of constraint G_2 were taken into consideration.

After discussing the question of the compatibility of the conditions of constraint in general, the author classifies these conditions for continuous systems into three groups. To the first belong the conditions which are determined by a single relation between the parameters defining the position of the system. The author distinguishes nine types in this class. To the second group, consisting of five types, belong the conditions which are determined by two rela-

* To this class belong inextensible strings and membranes, incompressible fluids and so on. The position of such systems cannot be determined by means of a finite number of parameters. But, although Mr. Painlevé considers in his lectures only systems the position of which can be determined by means of a finite number of parameters, the general propositions developed by him can be immediately extended to the class here mentioned.

tions between the parameters defining the system. Finally, the conditions determined by three relations form the third group and are all conditions of constraint without friction.

This general theory is followed by numerous applications worked out completely, and with the elegance and clearness which are characteristic of the two courses of lectures we have before us.

It is only fair in concluding this review to remark to Mr. Hermann's credit that the reading of these two volumes is not in the slightest degree trying to the eyes, which unfortunately could not be said with regard to, for instance, Mr. Klein's lithographed courses.

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January 24, 1896.

A GEOMETRIC PROOF OF A FUNDAMENTAL THEOREM CONCERNING UNICURSAL CURVES.

BY PROFESSOR W. F. OSGOOD.

1. IF $f(x,y)=0$ is the equation of an irreducible curve of deficiency 0, then, as is well known, the coördinates can be expressed as rational functions of a parameter λ:*

$$x=r_1(\lambda) \qquad y=r_2(\lambda)$$

where not only to a given value of λ corresponds one and only one point of the curve, but conversely to a given point (x,y) on the curve corresponds in general one and only one value of λ.† λ can be expressed as a rational function of x and y.

If a multiple-leaved Riemann surface spread out, say, over the x-plane be used to represent geometrically the above locus, $f(x,y)=0$, the deficiency of this surface will likewise be 0, and as is shown in the elements of Riemann's theory of functions,‡ there exist single-valued functions on such a surface having but a single pole, and that of the first order, and taking on every value once and only once on the surface. Call such a function λ. Being single-valued on the surface it will be a rational function of x and y: $\lambda=R(x,y)$,

*SALMON, *Higher Plane Curves*, p. 30; CLEBSCH-LINDEMANN, *Geometrie* vol. I., p. 883.

† CAYLEY has given to such curves the name *unicursal*.

‡ KLEIN, *Modulfunctionen*, vol. I., p. 493 et seq. PICARD, *Traité d' analyse*, vol. II., Ch. XVI.

and hence through this equation the multiple-leaved surface is transformed conformally into the single-leaved λ-plane. Hence x is a rational function of λ. And similarly for y.

2. Conversely, let the equations

$$\left.\begin{aligned} x&=r_1(\lambda)\\ y&=r_2(\lambda)\end{aligned}\right\} \qquad (1)$$

be given, where r_1 and r_2 are any rational functions of λ Then the locus of (x, y), if regarded as a plane curve, is an algebraic irreducible curve of deficiency 0; if represented by a multiple-leaved surface spread out over the x-plane, it gives rise to an algebraic surface consisting of a single piece, likewise of deficiency 0. (In connection with these statements see however the last lines of § 5.) But while to every value of λ there still corresponds one and only one point (x, y) of this surface, it is not true conversely that to a point (x, y) only one value of λ corresponds, as is shown by the example $x=\lambda^2$, $y=\lambda^3$. It is however always possible to replace λ by a new parameter μ, which is a certain rational function of λ:

$$\mu=r(\lambda)$$

such that x and y will be rational functions of μ, while to a given point (x, y) only one value of μ corresponds.*

3. The essential thought that underlies these theorems finds clear expression in the geometric method of conformal representation, and reciprocally the value of this method as a means of investigation is set in a strong light.

It is believed that it will not be without interest to the readers of the BULLETIN to see this simple example of the method worked out, and it is hoped furthermore that this example may be suggestive of a desirable form of presentation in instruction in the theory of functions.

4. Consider first the equation

$$x=r_1(\lambda)$$

To each value of λ corresponds one and only one value of x and the λ-surface consists of the single-leaved λ-plane. To each value of x correspond m values of λ, in general distinct, and the x-surface consists of an m-leaved Riemann surface connected through junctions along junction-lines. Call this surface T.

Now consider the second equation(1):

$$y=r_2(\lambda).$$

y will be a single-valued function on the surface T, since to

* An algebraic proof of this latter theorem was given by Lüroth, *Math Annalen*, vol. 9, 1875.

every point of T corresponds only one value of λ. But it may happen that for *every* value of x two or more of the m corresponding values of y coincide; so that the surface T would not represent y as a function of x.

Let us construct the surface that does represent y as a function of x. For this purpose first cut the leaves of T along the junction lines, thus obtaining m separate leaves, each containing one or more slits. Consider the first of these leaves. To each of its points corresponds a value of y, y_1, and it forms the first leaf of the surface to be constructed, S. Next let x leave this first leaf and crossing one of the slits, pass into a second leaf of T. Call the corresponding value of y in this second leaf y_2. Two cases can arise. Either y_2 is, in general, different from y_1 or y_2 is identical with y_1. In the first case this second leaf of T is to be connected with the first leaf along this slit and thus forms the second leaf of S. In the second case the two edges of the slit in the first leaf are to be joined with each other and this slit thus disappears from S;* the second leaf of T does not belong to S. The example already cited will serve here as illustration: $x=\lambda^2$, $y=\lambda^2$. The surface T consists of two leaves with junctions at 0 and ∞; the junction-line may be taken as the positive axis of reals. The points of the first leaf shall represent those pairs of values (x, λ) that flow continuously out of the pair $x=-1$, $\lambda=i$ when x describes this leaf; y_1 is thus equal to x. When x crosses the slit and thus passes into the second leaf of T, y_1 goes over into y_2. But $y_2=y_1$, both being equal to x, and hence in the construction of S this slit disappears from the first leaf. In this example this is the only slit in the first leaf and thus the surface that represents y as a function of x is the single-leaved x-plane.

Returning now to the general case, we treat the other slits in the first leaf, if there be any, in the same way and thus finally either all the slits are closed and the first leaf forms the whole surface S, or else this first leaf is connected with other leaves along at least one slit. Such leaves will surely belong to the surface S. These leaves may be already connected along their slits with the first leaf, as in the case

$$x=\lambda^6, \quad y=\lambda^3;$$

or perhaps it remained merely to join certain pairs of them

* It may happen that a slit, beside abutting on the two junctions at its extremities, passes through one or more junctions in its course. In that case I shall consider each of the segments of the slit as a separate slit.

along certain slits, as in the case

$$x=\lambda^6, \quad y=\lambda^2;$$

or it may be that one of these leaves contains a slit such that when the point(x, y) crosses this slit, y takes on still new values. Then these two leaves are to be joined along this slit and another leaf is added, as in the case

$$x=\lambda^6, \quad y=\lambda.$$

Proceeding in this way we finally obtain a closed surface S. This surface may or may not be the same as T. If it is, then it is of deficiency 0, being conformally related to the single-leaved λ-surface. λ is a single-valued function on this surface having no essentially singular points, and hence it is a rational function of x and y:*

$$\lambda=R(x, y).$$

Thus all the statements made in § 2 are proven.

5. If, however, only a part of the leaves severed from T have been incorporated into S, then it must be that in the construction of S as described above, a leaf L' was found in which the values of y were identical, for the same values of x, with values in another leaf L already belonging to S; and hence L' was not joined to S, the connection along the corresponding slit being between the leaves already in S.

Consider now the leaf L'. The values of y in this leaf are the same for the same values of x as in L. If we proceed to construct the Riemann surface for y as a function of x, starting with the leaf L', we shall obviously be led to a surface S' identical with S; thus S is repeated. If the leaves severed from T are not yet exhausted, it will be possible to construct from them still a third surface S'', likewise identical with S. And so on. Thus finally the leaves of T will have been combined in a set of ρ surfaces, S, S', $S''\ldots S^{(\rho-1)}$, each identical with the first, S. ρ will divide m. Let $m=m'\rho$. Then m' is the number of leaves in S. The examples already cited illustrate clearly what has just been said.

So much for the surface T and the surfaces S. Let us turn now to the corresponding regions of the λ-plane. To each of the m leaves of the surface T corresponds a region of the λ-plane, regions corresponding to leaves that are connected bordering on each other. If we recall the way in which the S-surface was generated by the successive addition of new leaves, we see that the counterpart of this process in the λ-plane consists in starting with the region

* HARKNESS and MORLEY, *Theory of Functions*, § 174; KLEIN-FRICKE, *Modulfunctionen*, vol. I., p. 499.

corresponding to the first leaf of S and then adding on bordering regions corresponding to the subsequent leaves of S, the final result being a connected region N, made up of m' of the former regions, which corresponds point for point and conformally to S. To S' will correspond a second such region N', to S'' a third, $N'',\cdots$to $S^{(\rho-1)}$, $N^{(\rho-1)}$, and these ρ regions will just cover the λ-plane.

Let λ_1 be any point lying in the region N. To λ_1 corresponds the point (x, y) of each of the surfaces $S, S',\cdots S^{(\rho-1)}$ and to these points correspond in turn values $\lambda_1', \lambda_1'',\cdots\lambda_1^{(\rho-1)}$ belonging respectively to $N', N'',\cdots N^{(\rho-1)}$. Let us from now on, discarding the surfaces $S', S'',\cdots S^{(\rho-1)}$, restrict ourselves to S, so that $\lambda'_1, \lambda_1'',\cdots\lambda_1^{(\rho-1)}$ will now be regarded as functions of (x, y) on S. Thus any region of S that does not contain any part of one of the slits that has been closed up or along whose edges the sheets have been connected differently in S from in T will be transformed conformally on a region of each of the domains $N, N',\cdots N^{(\rho-1)}$. If the point (x, y), moving on S, cross one of the slits just excepted, then, since this slit corresponds to the boundary between $N^{(i)}$ and $N^{(k)}$, $\lambda_1^{(i)}$ will cross this boundary and enter the region $N^{(k)}$, another one of the λ_1's entering at the same time the region $N^{(i)}$ just vacated by $\lambda_1^{(i)}$. And if (x, y) describe any closed path on S, the λ_1's will either describe closed paths or be permuted. Now form the symmetric product

$$\mu=\lambda_1\lambda_1'\lambda_1''\ldots\lambda_1^{(\rho-1)}. \tag{3}$$

This function is in general monogenic on S, since this is true of each of the factors (on account of the conformal correspondence), and furthermore it is single-valued. It has no essentially singular point and is therefore a rational function of x and y,

$$\mu=R(x,y). \tag{4}$$

The point $\lambda=\infty$ will lie in general entirely within one of the regions N, so that only one factor will ever become infinite, and this factor will afford a pole of the first order. No other factor becoming 0 simultaneously,* μ has but one pole and that of the first order, so that it takes on every value once and only once on S.

It thus appears that the surface S is transformed conformally on a single-leaved plane, the μ-plane. Hence we infer (1) that the deficiency of S is 0; (2) that x is a rational function of μ; for, because of the conformal relation between

* If these conditions, in a special case, should not happen to be fulfilled, λ may be replaced through a suitable linear transformation by $\overline{\lambda}$, so that they will now be fulfilled.

the surface S and the μ-plane, x behaves in every region of the μ-plane like a rational function; (3) that y is likewise a rational function of μ; for the Riemann surface that represents x as a function of y is conformally related to S and hence to the μ-plane, and y behaves therefore in every region of the μ-plane like a rational function; (4) that μ is a rational function of λ; this can also be inferred easily from the conformal representation, but as it follows immediately from equations (1) and (4), we will not insist on this method.

Thus all of the statements of § 2 have been proven.

Example.
$$x=\frac{(\lambda^4+\lambda^{-4}+14)^3}{108\ (\lambda^4+\lambda^{-4}-2)^2}=\frac{4\,(\mu^2-\mu+1)^3}{27\ (\mu-1)^2}$$

$$y=-\left(\frac{\lambda^2-1}{2\lambda}\right)^2=\mu$$

$$\mu=-\tfrac{1}{4}\,(\lambda-\lambda^{-1})^2$$

See Klein-Fricke, *Modulfunctionen*, vol. I. p. 75, for the division of the λ-plane. The $\rho=4$ regions N,N',N'',N''' appear in the figure on p. 80. The notation λ, μ is there just the reverse of that in this paper.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., December, 1895.

NOTES ON THE EXPRESSION FOR A VELOCITY-POTENTIAL IN TERMS OF FUNCTIONS OF LAPLACE AND BESSEL.

BY PROFESSOR JAMES McMAHON.

1. *Differential equation for ψ.* The partial differential equation to be satisfied by a velocity-potential in an elastic fluid is, in rectangular coördinates,*

$$\frac{\delta^2\psi}{\delta x^2}+\frac{\delta^2\psi}{\delta y^2}+\frac{\delta^2\psi}{\delta z^2}=\frac{1}{a^2}\frac{\delta^2\psi}{\delta t^2}, \qquad (1)$$

in which a^2 = pressure/ density, and $\psi\ (x, y, z, t)$ is a function whose derivatives as to x, y, z give the velocity-components of the fluid particle that occupies the position (x, y, z) at time t.

2. *Particular solution in polar coördinates.* When (1) is transformed to polar coördinates r, θ, φ, it can, as shown in

*RAYLEIGH, *Theory of Sound*, vol. II., p. 15.

standard works,* be satisfied by a product-function of the form

$$\psi_n = r^{-\frac{1}{2}} J_{\pm(n+\frac{1}{2})}(kr).\ S_n(\theta, \varphi).\ \begin{matrix}\sin\\ \cos\end{matrix} kat, \qquad (2)$$

in which S_n is any surface harmonic of order n, and $J_{\pm(n+\frac{1}{2})}$ is a Bessel function of order $\pm(n+\frac{1}{2})$. Here ψ_n stands for the part of ψ that involves nth order harmonics, and may be written in a more explicit form as the sum of the four terms implied in (2); thus

$$\begin{aligned} r^{\frac{1}{2}}\psi_n = {} & J_m(kr)\ [S_n \sin kat + S'_n \cos kat] + \\ & J_{-m}(kr)\ [S''_n \sin kat + S'''_n \cos kat], \end{aligned} \qquad (3)$$

where $m=n+\frac{1}{2}$, and the symbols S_n represent arbitrary nth order harmonics. The conditions of a particular problem may impose restrictions on these coefficients; and the chief object of this paper is to show how to choose the relations between the symbols S_n in order to adapt equation (3) to some of the typical problems in fluid motion to which the functions of Laplace and Bessel are applicable.† In all cases the complete value of ψ is to be built up by putting $n=0, 1, 2, \ldots$, and determining the arbitraries by the initial distribution of condensation and velocity.

3. *Phase angles.* It will be sometimes convenient to put equation (3) in the form

$$r^{\frac{1}{2}}\psi_n = T_n J_m(kr)\sin(kat+\varepsilon_n) + T'_n J_{-m}(kr)\sin(kat+\varepsilon'_n), \qquad (4)$$

wherein

$$T_n \cos\varepsilon_n,\ T_n \sin\varepsilon'_n,\ T_n\cos\varepsilon'_n,\ T'_n \sin\varepsilon'_n$$

take the place of S_n, S'_n, S''_n, S'''_n; the phase angles $\varepsilon_n, \varepsilon'_n$, being thus functions of θ, φ.

4. *Lemma.* To find the relations between the phase angles $\varepsilon_n, \varepsilon'_n$, and between the coefficients T_n, T'_n, in order that the motion of the medium at a great distance from the origin may be approximately represented by a single wave, either diverging from the pole or converging towards it:

When r is large put

$$J_{\pm m}(kr) = r^{-\frac{1}{2}} \cos(kr - \tfrac{1}{4}\pi \pm \tfrac{1}{2} m\pi),\ ‡$$

dropping a numerical factor; substitute in (4), and change trigonometric products into sums, then

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*Gray and Matthews, *Bessel Functions*, pp. 213, 215. *Theory of Sound*, Vol. II., pp. 205, 229, 230, in which the J functions are replaced by two other particular solutions of the equation that $r^{-\frac{1}{2}}J$ satisfies.

† The advantage of the form of solution in equation (3) is that it gives a "realized" result in terms of known functions and without restriction of phase.

‡ Bessel Functions, p. 40.

$$r\phi_n = T_n \sin\left[k(at+r)+\varepsilon_n-\tfrac{1}{4}\pi-\tfrac{1}{2}m\pi\right]+ T'_n \sin\left[k(at+r)+\varepsilon'_n-\tfrac{1}{4}\pi+\tfrac{1}{2}m\pi\right] \quad (5)$$

$$+ \text{ terms involving } at-r.$$

This expresses the superposition of a divergent and a convergent wave, represented by the terms in $at-r$, $at+r$, respectively. In order that the latter wave may be absent, the corresponding terms indicated in (5) must cancel each other for all values of the variables. This requires that $T_n = T'_n$, and also that

$$(\varepsilon'_n+\tfrac{1}{2}m\pi)-(\varepsilon_n-\tfrac{1}{2}m\pi) = \text{an odd multiple of } \pi,$$

say congruent with $(2n+1)\pi$, *i. e.*, with $2m\pi$,

$$\therefore \varepsilon'_n-\varepsilon_n = m\pi, \text{ or a congruent.} \quad (6)$$

Similarly the divergent wave is absent when

$$T_n = T'_n, \quad \varepsilon_n-\varepsilon'_n = m\pi.$$

Thus the velocity potential of a wave which at a great distance from the origin is $\left\{\begin{matrix}\text{divergent}\\ \text{convergent,}\end{matrix}\right.$ consists of terms of the form

$$r^{\frac{1}{2}}\varphi_n = T_n[J_m(kr)\sin(kat+\varepsilon_n)+J_{-m}(kr)\sin(kat+\varepsilon_n\pm m\pi)]. \quad (7)$$

5. *Relations between the symbols S_n, for divergence or convergence.* Put, according to Art. 3,

$$S_n = T_n \cos\varepsilon_n, \qquad S'_n = T_n \sin\varepsilon_n,$$

$$S'_n = T_n \cos(\varepsilon_n \pm m\pi), \quad S''_n = T_n \sin(\varepsilon_n \pm m\pi),$$

whence the required relations are

$$\therefore \quad S'_n = \mp S'_n \sin m\pi, \quad S''_n = \pm S_n \sin m\pi. \quad (8)$$

6. *Resolution of ϕ_n.* The expression in (3) can be exhibited as the sum of two parts representing divergent and convergent waves. For write

$$r^{\frac{1}{2}}\phi_n = T_n[J_m(kr)\sin(kat+\varepsilon_n)+J_{-m}(kr)\sin(kat+\varepsilon_n+m\pi)]+$$

$$T'_n[J_m(kr)\sin(kat+\varepsilon'_n)+J_{-m}(kr)\sin(kat+\varepsilon'_n-m\pi)], \quad (9)$$

the first part representing a divergent wave, and the latter a convergent one, by (7); compare with (3), equate coefficients, and solve; then T_n, T'_n, ε_n, ε'_n are determined by

$$2T_n\cos\varepsilon_n = S_n+S''_n\sin m\pi, \quad 2T_n\sin\varepsilon_n = S'_n-S''_n\sin m\pi,$$

$$2T'_n\cos\varepsilon'_n = S_n-S''_n\sin m\pi, \quad 2T'_n\sin\varepsilon'_n = S'_n+S''_n\sin m\pi. \quad (10)$$

7. *Application to spherical vibrator.* The surface of a sphere of radius c is maintained in vibration with given frequency, and the normal surface-velocity is a given harmonic function of latitude and longitude of the form

$$v_n(\theta, \varphi, t) = \bar{S}_n \sin kat + \bar{S}'_n \cos kat, \tag{11}$$

to find the velocity potential at any point (r, θ, ϕ) of the medium at any time t:

Use (3), find the radial velocity-function $\delta\psi_n/\delta r$, and put $r=c$, then

$$v_n = \left(\frac{\delta \psi_n}{\delta r}\right)_{r=c} = F_n(kc)\,[S_n \sin kat + S'_n \cos kat] + F_{-n}(kc)\,[S''_n \sin kat + S'''_n \cos kat], \tag{12}$$

$$\text{in which} \qquad F_n(kr) = \frac{\delta}{\delta r} \cdot r^{-\frac{1}{2}} J_n(kr).$$

Identifying (11),(12) gives two equations, and the conditions for divergence in (8), furnish two others to determine S_n, S'_n, S''_n, S'''_n in terms of $\bar{S}_n$, $\bar{S}'_n$; thus

$$\gamma S_n = \alpha \bar{S}_n + \beta \bar{S}'_n, \quad S''_n = -S'_n \cos n\pi,$$
$$\gamma S'_n = -\beta \bar{S}_n + \alpha \bar{S}'_n, \quad \bar{S}'''_n = S_n \cos n\pi, \tag{13}$$

where $\alpha = F_n(kc)$, $\beta = F_{-n}(kc) \cos n\pi$, $\gamma = \alpha^2 + \beta^2$; and these values of the symbols S_n substituted in (3) give the required velocity-potential at an external point.*

The internal potential is of different form; for the disturbed region now includes the point $r=0$, at which $J_{-n}(kr)$ becomes infinite, hence the arbitrary coefficients of this function must be zero. Putting $S''_n = \bar{S}'''_n = 0$, and identifying (11), (12), determines S_n, S'_n, and gives

$$r^{\frac{1}{2}}\psi_n = \frac{1}{F_n(kc)} J_n(kr) \left[\bar{S}_n \sin kat + \bar{S}'_n \cos kat\right]. \tag{14}$$

8. *Free vibrations inside a fixed spherical envelope.* Here, as in the latter part of Art. 7, the form is

$$r^{\frac{1}{2}}\psi_n = J_n(kr)\left[S_n \sin kat + S'_n \cos kat\right], \tag{15}$$

but the possible values of the free periods, which depend on k, are to be found from the equation $F_n(kc) = 0$, *i. e.*

$$\frac{\delta}{\delta r}\left[r^{-\frac{1}{2}} J_n(kr)\right]_{r=c} = 0;\dagger \tag{16}$$

*This solution differs from that given in Theory of Sound, vol. II., p. 207, by being expressed in "realized" form by means of Bessel functions, and by not restricting v_n to be in the same phase over the surface $r=c$.

† *Annals of Mathematics*, vol. 9, No. 1, p. 27.

while the symbols S_n are determined by the harmonic elements of the initial distribution of velocity and condensation.*

9. *Free vibrations between two concentric spherical surfaces.* Since the radial velocity at the surface $r = r_1$ is zero, then

$$F_m(kr_1)\left[S_n \sin kat + S'_n \cos kat\right] + F_{-m}(kr_1)\left[S''_n \sin kat + S'''_n \cos kat\right] = 0;$$

and there is a similar equation involving r_2.

These must be satisfied for all values of θ, φ, t,

$$\therefore S_n F_m(kr_1) = -S''_n F_{-m}(kr_1); \; S'_n F_m(kr_1) = -S'''_n F_{-m}(kr_1),$$

with two similar equations in r_2,

$$\therefore \frac{S''_n}{S_n} = \frac{S'''_n}{S'_n} = -\frac{F_{-m}(kr_1)}{F_m(kr_1)} = -\frac{F_{-m}(kr_2)}{F_m(kr_2)} = \rho, \text{ say.} \quad (17)$$

The possible values of k, and of the wave length $2\pi/k$, are to be found from the third of these equalities ;† and then S''_n, S'''_n are known multiples of S_n, S'_n. Thus (3) takes the form

$$r^{-\frac{1}{2}}\psi_n = [J_m(kr) + \rho J_{-m}(kr)]\,(S_n \sin kat + S'_n \cos kat), \quad (18)$$

an equation which, extended to the whole of space, gives a series of nodal spherical surfaces, of which $r = r_1$, and $r = r_2$ are a pair. At such surfaces the superposed divergent and convergent waves interfere.

ADDITIONAL NOTE ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

In a previous note (pp. 72–75) it has been shown that every divergent series oscillating between finite limits can by a proper arrangement of its terms be made convergent. We will now extend those results to the case when one or both limits between which the series oscillates are infinite. To this end it suffices to consider, together with *regular* se-

* The work is exemplified for the case $n = 1$, *Theory of Sound*, pp. 236, 237.

† *Annals of Mathematics*, vol. 9, No. 1, pp. 29, 30.

quences of numbers, such sequences as tend to infinity *in a determinate way*. The number N_i then may be $+\infty$ or $-\infty$ and the infinite series

$$a_{1,i} + a_{2,i} + \cdots + a_{n,i} + \cdots$$

of Theorem I. (p. 73) then tends to infinity *in a determinate way*. This series is *unconditionally divergent*. A similar change in the considerations of Theorem II. (p. 74) shows that the series

$$(u_1 + u_2 + \cdots + u_{\mu_{1,1}}) + (u_{\mu_{1,1}+1} + \cdots + u_{\mu_{k,1}}) + \cdots$$

tends to infinity in a determinate way, but it must be remembered that here each expression in parenthesis is to be considered as a *single* term, and that this determinateness would be lost at once if the brackets were dropped.

Example: $1-2+3-4+5-\cdots$

$g=2$; $N_1 = +\infty = 1+(-2+3)+(-4+5)+\cdots$

$N_2 = -\infty = (1-2)+(3-4)+\cdots$

We will say that an infinite series tends to infinity only if it does so in a determinate way. It was also in this sense that the expression was used in Theorem III. (p. 75). We may now add

Theorem VI. Every oscillating series can by a proper arrangement of its terms be made convergent or tending to infinity.

To conclude, a remark must be made with regard to the different rearrangements of the terms of an infinite series. On p. 75 it was mentioned that by a proper arrangement, both commutative and associative, of the terms of a conditionally divergent series, this series can be made to converge to any arbitrarily assigned number. It is important to take into consideration whether the associative change precedes the commutative or whether the inverse takes place. Of course the remark just mentioned applies only to series such that $\lim_{m=\infty} (u_m) = 0$, if u_m be the general term of the series, this condition being implicitly assumed in the extension of Riemann's proposition to conditionally divergent series.

Consider now a divergent series as given in Theorem II. (p. 74) and let again

$$N_i = (u_1 + \cdots + u_{\mu_{1,1}}) + (u_{\mu_{1,1}+1} + \cdots + u_{\mu_{k,1}}) + \cdots$$

Here each expression in parenthesis is to be regarded as a *single* term, and the series may be either absolutely convergent or only semi-convergent. It seems, therefore, that we were not justified in saying (Theorem V.) that every conditionally

divergent series can by a proper arrangement of its terms be made *semi-convergent;* that we should have used the word *convergent* instead. But the character of semi-convergence appears at once if we reverse the order of our changes in the original series, *i. e.*, if we first introduce a commutative and then proper associative changes. An oscillating series having necessarily an infinite number of positive and negative terms, the numbers N_i of Theorem II. will in general change their values if the order of terms in the given oscillating series be changed. Indeed, provided $\lim_{n=\infty}(u_n)=0$, it is possible by a proper commutative arrangement to make not only the N_i but also their number g quite arbitrary. This is why we have used the word semi-convergent, and not convergent, in Theorem V.

As an example, let us take again the series (2) of the first note:

$$\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{2}{3}-\tfrac{3}{4}+\tfrac{3}{4}-\cdots$$

which may be written as follows:

$$\sum_1^\infty\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}+\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

If we now introduce the following communicative change

$$\sum_1^\infty\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

we obtain $g=3$ and

$$N_1=\tfrac{1}{2}=\tfrac{1}{2}-\sum_1^\infty\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

$$N_2=-\tfrac{1}{2}=\sum_1^\infty\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

$$N_3=-\tfrac{3}{2}=\tfrac{1}{2}-\tfrac{2}{3}-\tfrac{3}{4}+\sum_1^\infty\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n+2}{2n+3}-\frac{2n+3}{2n+4}\right\}$$

Thus the particular change here introduced has increased the number g by unity. It is easy to apply a number of other modifications by following the same process.

NOTES.

AT the Lübeck meeting of the German Mathematical Society, held in September, 1895, it was decided to combine in one volume the official reports of the Vienna and Lübeck meetings. This will form vol. IV of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Berlin, Reimer). Through the courtesy of the executive committee we have received advance sheets of the first twenty pages of this volume, containing the general reports of the secretaries, but not the abstracts of the papers.

It appears from the general report of the Vienna meeting (September, 1894) that the following papers should be added to the partial list given in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, vol. I. (1894-95), pp. 78-79:

GUTZMER (Berlin): "A new derivation of Kirchhoff's expression for Huygens's principle."

KIEPERT (Hannover): "On the mathematical preparation desirable for specialists in assurance."

KRAUSE (Dresden): "On the theory of the transformation of the elliptic functions."

LANDSBERG (Heidelberg): "On the theory of algebraic numbers."

LERCH (Prague): "On an integral occurring in Cauchy's transformation of the elementary elliptic function of the third kind."

J. SCHMIDT (Budapest): "Remarks on Bolyai's Tentamen."

WÄLSCH (Prague): "On a method of treating surfaces of the third order."

WANGERIN (Halle): "On the history of the theory of conformal transformation."

THE papers originally announced by Nagy, Peschka, Schapira, Schlesinger, and Studnička (loc. cit.) were not read.

At the Lübeck meeting the following papers were read:

HILBERT (Göttingen) and MINKOWSKI (Königsberg): Account of their special report on the present state of the theory of numbers.

E. KÖTTER (Berlin): Account of his special report on the development of synthetic geometry.

FREGE (Jena): "On Prof. Peano's and my own symbolic algebra (Begriffsschrift)."

WANGERIN (Halle): "On Franz Neumann's mathematical works."

LAMPE (Berlin): "On the compilation of a general bibliography."

HEFFTER (Giessen): "On common divisors and common multiples of linear differential expressions."

VOSS (Würzburg): "On infinitesimal deformations of surfaces."

POKROVSKY (Kiev): "On the hyperelliptic functions of two arguments."

SOUSLOV (Kiev): "On a continuous group of Darboux rotations."

JUKOVSKY (Moscow): "Geometrical interpretation of S. Kovalevsky's case of the motion of a heavy rigid body about a fixed point."

FRICKE (Braunschweig): "The regions of discontinuity of the groups of real linear substitutions of a complex variable."

KLEIN (Göttingen): "On the theory of ordinary continued fractions."

GORDAN (Erlangen): "Pascal's theorem."

SCHUBERT (Hamburg): "Correlative correspondence in n dimensions."

GUTZMER (Berlin): "On certain linear differential equations."

GODT (Lübeck): "On the Feuerbach circle and the Steiner curve of fourth order and third class."

KOHN (Vienna): "On the geometric interpretation of homogeneous coördinates."

LONDON (Breslau): "On cubic constructions."

MEYER (Clausthal): "On L. Schendel's memoir on infinite series and products."

GUTZMER (Berlin): "On Schoute's paper on certain envelopes."

SCHÜTZ (Göttingen): "On an interrelated group of thermodynamic, electrodynamic and astrophysical facts."

SOMMERFELD (Göttingen): "Diffraction problems in exact treatment."

The special report on the present state of the theory of numbers, by Hilbert and Minkowski, will probably be published before the end of the present year; E. Kötter's special report on the development of synthetic geometry is expected to follow soon.

Plans for a number of other reports on special branches of mathematics were discussed at both meetings. Thus Pringsheim is preparing a report on the theory of infinite series, Czuber on the calculus of probabilities, Stäckel on differential geometry, Wälsch on line geometry. It was re-

solved at Lübeck that Wirtinger's report on Abelian functions should be postponed till after the publication of the third volume of Weierstrass' collected works and of his lectures on Abelian functions. At the same time, the theory of the functions of a real argument was designated as a subject on which a report would be desirable.

The project of a mathematical dictionary has taken somewhat more definite form, the academies at Vienna, Munich and Göttingen having promised financial assistance; B. G. Teubner will be the publisher.

The Society expressed its sympathy with the idea of an International Congress of Mathematicians, but declined to take the initiative in the matter. A communication from Prof. Molenbroek (Hague), concerning the formation of an international association for the promotion of the study of quaternions and allied subjects, was not received very favorably, the Society deeming it unwise to form such an association merely for promoting a very restricted branch of mathematical science.

Both at Vienna and Lübeck the German Mathematical Society held its sessions together with the section for mathematics and astronomy of the Society of German Naturalists and Physicians. At Lübeck a combined session with the physicists was arranged for. Besides the last two of the above-named papers, a report by Prof. HELM (Dresden), "On the present state of energetics," was read at this session. It provoked a prolonged and thorough discussion, but will not be included in the *Jahresbericht*.

A committee, consisting of Professors Brill, Wangerin and Gutzmer, was appointed to draft and present an address of congratulation to Prof. Weierstrass on his eightieth birthday (31st October, 1895).

The officers of the Society are at present: A. Brill, President; A. Gutzmer, Secretary and Treasurer; A. Wangerin and A. Gutzmer, Publication Committee; F. Klein and G. von Escherich, members of the Council. The membership is 273.

AMONG the prizes awarded at the annual meeting of the French Academy of Sciences were the following: *Prix Francœur* (1,000 francs), awarded annually for discoveries or works useful to the progress of mathematical science, to J. Andrade; *Prix Poncelet* (2,000 francs), awarded annually for the work, published during the preceding ten years, which, in the judgment of the Academy, has been most useful to the progress of mathematical science, to G. Robin for the

whole of his work on mathematical physics; *Prix Petit d'Ormoy* (10,000 francs), awarded every second year, for works on pure or applied mathematics, to the late Albert Ribaucour.

The subjects set by the Academy for the next mathematical prizes are as follows: *Grand Prix des Sciences mathématiques* (10,000 francs), to advance in some important respect the algebraic theory of groups of substitutions of n letters, papers to be submitted by October 1, 1896. *Prix Bordin* (3,000 francs), to advance in some important respect the theory of geodesic lines, the case of a linear element with any number of variables not being excluded, papers to be submitted by October 1, 1896. *Prix Damoiseau* (1,500 francs); subject in 1896, to connect, by means of the theory of perturbations, the different appearances of Halley's comet, going back to Toscanelli's observations in 1456 and taking into account the attraction of Neptune, and to calculate with precision the next return of this comet in 1910; subject in 1898, to give an exposition of the theory of the perturbations of Saturn's satellite Hyperion, discovered in 1848 simultaneously by Bond and Lassell, taking into account principally the action of Titan, to compare the observations with the theory, and to deduce therefrom the value of the mass of Titan. Papers for the *Prix Damoiseau* must be submitted by June 1.

THE UNIVERSITY OF CHICAGO: SUMMER 1896. The following mathematical courses will be offered: By Professor Moore, Theory of Numbers, Differential Equations (with introduction to Lie's continuous transformation-groups); by Professor Bolza, Theory of Substitutions, Theory of Functions of a Complex Variable; by Professor Miller, of the University of Indiana, Analytic Geometry of Three Dimensions; by Dr. Young, Conferences on Mathematical Pedagogy, Theory of Equations, College Algebra; by Mr. Slaught, Advanced Integral Calculus, Introductory Course in Differential and Integral Calculus; and by Mr. Baker, Analytic Geometry of the plane. The pedagogical conferences are two hours weekly for six weeks, and the other courses are four or five hours weekly for twelve weeks from July 1, 1896. Those who expect to work in mathematics at the University of Chicago during the coming summer, as as well as those who desire further information, are requested to communicate with Professor Moore.

THE 1896 volume of the *Annuaire*, published by the Bureau des Longitudes, has been issued. It contains the usual mass

of material devoted to astronomical and other science. Among the appendices are two articles by M. Cornu, upon forces acting at a distance, and upon Fresnel's optical work. There is also a report by M. Janssen about his third ascent of Mont Blanc. The list of members of the Bureau contains the names of two Americans: Dr. B. A. Gould and Mr. G. Davidson.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Untersuchungen über die Reihe $1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \ldots\ldots$ (1826.) Herausgegeben von A. Wangerin. (Ostwald's Klassiker der exakten Wissenschaften, No. 71.) Leipzig, Engelmann, 1896. 8vo. 46 pp. Cloth. Mk. 1.00

COCULESCO (N.). Sur les expressions approchées des termes d'ordre élevé dans le développement de la fonction perturbatrice. Paris, Gauthier-Villars, 1895. 4to. 90 pp.

CONANT (L. L.). The number concept; its origin and development. New York, Macmillan, 1896. 12mo. Cloth. $2.00

DEWEY (J.). See MCLELLAN (J. A.) and DEWEY (J.).

EPSTEIN (P.). Zur Lehre von den hyperelliptischen Integralen. [Diss.] Strassburg, 1895. 4to. 58 pp.

GÖPEL (A.). Entwurf einer Theorie der Abel'schen Transcendenten erster Ordnung. (1847.) Herausgegeben von H. Weber. Aus dem Lateinischen übersetzt von A. Witting. (Ostwald's Klassiker der exakten Wissenschaften, No. 67.) Leipzig, Engelmann, 1896. 8vo. 60 pp. Cloth. Mk. 1.00

KLEIN (F.). Ueber die Arithmetisirung der Mathematik. [*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1895, Heft 2.] 8vo. 10 pp.

———. Ueber eine geometrische Auffassung der gewöhnlichen Kettenbruchentwickelung. [*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1895, Heft 3.] 8vo. 3 pp.

MCLELLAN (J. A.) and DEWEY (J.). The psychology of number and its applications to the methods of teaching arithmetic. (International education series, No. 23.) New York, Appleton, 1895. 8vo. 12 and 310 pp. Cloth. $1.50.

MATTHIESSEN (L.). Grundzüge der antiken und modernen Algebra der litteralen Gleichungen. (1878.) 2te Ausgabe (unchanged, but reduced in price). Leipzig, Teubner, 1896. 8vo. 16 and 1,002 pp. Mk 8.00

MOLK (J.). See TANNERY (J.) and MOLK (J.).

NEPPI-MODONA (A.) e VANNINI (T.). Questioni e formole di geometria analitica ad una e due dimensioni. Palermo, 1895. 8vo. 320 pp. Fr. 7.50

PASCAL (E.). Teoria delle funzioni ellittiche. Milano, Hoepli, 1896. 16mo. 12 and 228 pp.

POINCARÉ (H.). Calcul des probabilités. Leçons professées au cours de physique mathématique pendant le 2e semestre 1893–94, rédigées par A. Quiquet. Paris, Carré, 1896. 8vo. 280 pp.

PUCHBERGER (E.). Eine allgemeinere Integration der Differentialgleichungen. Heft 3. Wien, Gerold, 1896. 8vo. 6 and 52 pp. Mk. 1.60.

TANNERY (J.). et MOLK (J.). Éléments de la théorie des fonctions elliptiques. (En 4 volumes.) Vol. II: Calcul différentiel, partie 2. Paris, Gauthier-Villars, 1896. 8vo. 6 and 300 pp. Fr. 9.00

TORRES (L.). Memoria sobre las maquinas algébricas. Madrid, 1895. 4to. 108 pp. Fr. 4.00

VANNINI (J.). See NEPPI-MODONA (A.) e VANNINI (J.).

VELDE (A.). Ueber die Curven, deren Bogen der Tangente des Lichtstrahlwinkels proportional ist, und die damit verwandten Curvenscharen. [Diss.] Halle, 1895. 8vo. 42 pp., 2 plates.

WANGERIN (A.). See ABEL (N. H.).

WEBER (H.). See GÖPEL (A.).

WITTING (A.). See GÖPEL (A.).

II. ELEMENTARY MATHEMATICS.

ESMARCH (B. K.). Die Kunst des Stabrechnens. Gemeinfassliche und vollständige Anleitung zum Gebranche des Rechenstabes auf allen Gebieten des praktischen Rechnens. Leipzig, Günther, 1896. 8vo. 192 pp., 2 plates. Cloth. Mk. 4.00

GELIN (E.). Précis de trigonométrie rectiligne. Namur, 1895. 8vo. 72 pp. Fr. 1.50

HOLMAN (S. W.). Computation rules and logarithms, with tables of other useful functions. New York and London, Macmillan, 1896. 8vo. 14 and 74 pp. Cloth. $1.00

KRÖGER (M.). Leitfaden für den Geometrie-Unterricht. 10te Auflage. Hamburg, Meissner, 1895. Mk. 1.00

LODGE (A.). Mensuration for senior students. London and New York, Longmans, 1895. 12mo. 14 and 274 pp. $1.50

MEIGEN (F.). Lehrbuch der Geometrie. (Technische Lehrhefte, Mathematik, Heft 4.) Hildburghausen, Pezoldt, 1895. 8vo. 4 and 82 pp. Mk. 2.00

NELL (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Functionen, nebst den Logarithmen für Summe und Differenz zweier Zahlen, deren Logarithmen gegeben sind, sowie einigen anderen Tafeln. 8te Auflage. Darmstadt, 1895. 8vo. 20 and 104 pp. Cloth. Mk. 1.50

PINCHERLE (S.). Algebra elementare. 6a edizione. Milano, Hoepli, 1896. 16mo. 8 and 210 pp.

——— Esercizî sull' algebra elementare. Milano, Hoepli, 1896. 24mo. 135 pp.

SCHULTZ (E.). Leitfaden der Planimetrie für Werkmeisterschulen und gewerbliche Fortbildungsschulen. Teil 2. Essen, Baedeker, 1896. 8vo. 4 and 66 pp. Cloth. Mk. 0.75

SYLWAN (O. C.). Elementen i Algebran. Stockholm, 1895. 8vo. 108 pp. Mk. 1.80

WENTWORTH (G. A.). Syllabus of geometry. Boston, Ginn, 1896. 16mo. 2 and 50 pp.

III. APPLIED MATHEMATICS.

ALHEILIG et ROCHE (C.). Traité des machines à vapeur, rédigé conformément au programme du cours de machines à vapeur de l'École centrale. (En 2 volumes.) Vol. II. Paris, Gauthier-Villars, 1896. 8vo. 12 and 606 pp. Fr. 18.00

BEHSE (W. H.). Die darstellende Geometrie für Real-, Gewerbe- und Werkmeisterschulen, sowie zum Selbstunterricht für Bautechniker und Mechaniker. Nach dessen Tode bearbeitet von P. Berthold. Teil 1: Die Projektionslehre; Konstruktion der Durchschnittsfiguren; windschiefe Flächen; Spirallinien und Spiralflächen; schräge Projektion. 5te Auflage. Leipzig, Arnd, 1896. 8vo. 8 and 150 pp. Mk. 3.00

BERTHOLD (P.). See BEHSE (W. H.).

BOLTZMANN (L.). See MAXWELL (J. C.).

BORCH (S. C.). Forelaesninger over Maskinlaere. 2te udgave. Del II.: Maskindeles Beregning og Konstruktion; Arbejdsmaskiner. Kjöbenhavn, 1895. 8vo. 448 pp. 16 plates. Mk. 10.00.

BOVEY (T. H.). A treatise on hydraulics. New York, Wiley, 1895. 8vo. 8 and 338 pp. Cloth. $4.00.

COTTERILL (J. H.). The steam engine considered as a thermodynamic machine. A treatise on the thermodynamic efficiency of steam engines. 3d edition, revised. London, Spon, 1896. 8vo. 444 pp. 15s.

DU BOIS (A. J.). The elementary principles of mechanics. Vol. III: Kinetics. New York, Wiley, 1895. 8vo. 10 and 296 pp. Cloth. $3.50.

ENGESTRÖM (F. VON). Hvad kunna och vilja Lifförsäkringsbolagen. Kort redogörelse för Lifförsäkringens teoretiska Grunder och praktiska Möjligheter. Helsingfors, 1895. 8vo. 64 pp. Mk. 4.00.

FOUSSEREAU (G.) Leçons de physique à l'usage des élèves de la classe de mathématiques spéciales. (En 3 volumes.) Vol. I: Optique. Paris, 1895. 8vo. Fr. 13.50

FRANKLIN (W. S.). See NICHOLS (E. L.) and FRANKLIN (W. S.)

MAGGI (G. A.). Principî della teoria matematica del movimento dei corpi; corso di meccanica razionale. Milano Hoepli, 1896. 8vo. 18 and 504 pp. Fr. 12.00

MAXWELL (J. C.). Ueber Faraday's Kraftlinien. (1855.) Herausgegeben von L. Boltzmann. (Ostwald's Klassiker der exakten Wissenschaften, No. 69.) Leipzig, Engelmann, 1896. 8vo. 130 pp. Cloth. Mk. 2.00

NEUMANN (C.). Allgemeine Untersuchungen über das Newton'sche Princip der Fernwirkungen, mit besonderer Rücksicht auf die elektrischen Wirkungen. Leipzig, Teubner, 1896. 8vo. 22 and 292 pp. Mk. 10.00

NICHOLS (E. L.) and FRANKLIN (W. S.). The elements of physics. A college text-book. (In 3 volumes.) Vol. I.: Mechanics and heat. New York, Macmillan, 1896. 8vo. 228 pp. Cloth. $1.50

PLÜCKER (J.). Gesammelte wissenschaftliche Abhandlungen. Im Auftrag der K. Gesellschaft der Wissenschaften in Göttingen herausgegeben von A. Schoenflies und F. Pockels. (In 2 Bänden.) Vol. II.: Physikalische Abhandlungen, herausgegeben von F. Pockels. Leipzig, Teubner, 1896. 8vo. 18 and 834 pp., 9 plates. Mk. 30.00

POCKELS (F.). See PLÜCKER (J.).

ROCHE (C.). See ALHEILIG et ROCHE (C.).

ROSENBURG (F.). First stage mechanics. For the examination of the Science and art department. London, Clive, 1895. 8vo. 304 pp. 2s.

SANDRUCCI (A.). Le teorie sull' efflusso dei gas e gli esperimenti di G. A. Hirn. Firenze, Minori, 1895. 8vo. 60 pp.

WEILER (A.). Neue Behandlung der Parallelprojektionen und der Axonometrie. (1889.) 2te Ausgabe (unchanged, but reduced in price). Leipzig, Teubner, 1896. 8vo. 8 and 210 pp. Mk. 2.80